A Theory of Electoral Equilibrium: A Spatial Analysis Based on the Theory of Games

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Two areas of political science that utilize rigorous deductive formulations are: (1) the paradox of voting, and (2) spatial analysis of election competition. It is not surprising that the analysis of the paradox and the development of spatial theory occur simultaneously because the concerns of both endeavors are parallel: we study the paradox to ascertain conditions under which majority-rule equilibria exist, that is, conditions under which at least one alternative exists that cannot be

defeated by any other alternatives in a majority vote;[1] we study spatial theory to ascertain from citizen preferences and decision rules the strategies candidates should adopt. To identify such strategies, however, we must ascertain whether or not an electoral equilibrium exists, which, in the case of competition between two plurality-maximizing candidates, is equivalent to ascertaining whether or not a majority-rule equilibrium exists.[2]

This correspondence reveals a constraint on the development of spatial theory. The existing sufficient conditions for a majority-rule equilibrium if the domain of social choice is a single issue (unidimensional), require that citizens have single-peaked preferences. If we define the domain of social choice over many issues (multidimensional), then present models require, typically, that the citizens' most preferred positions be distributed symmetrically about the mean preference and that citizens' utility functions also satisfy certain symmetry properties. We render spatial models more consonant with reality: by moving from a unidimensional to a multidimensional model; by allowing citizens to abstain; by considering alternative preference distributions; and by permitting candidates to maximize either plurality or votes. We are not certain, though, that an equilibrium exists if our combinations of assumptions do not satisfy any of these sufficient conditions. Hence, those theorems of spatial theory that demonstrate the existence of

[1] See especially Duncan Black, *The Theory of Committees and Elections* (Cambridge: At the University Press, 1962); Charles R. Plott, "A Notion of Equilibrium and Its Possibility Under Majority Rule," *American Economic Review*, 57 (September 1967), 787–806; A. K. Sen, "A Possibility Theorem on Majority Decisions," *Econometrica*, 34 (April 1966), 491–499; and Gordon Tullock, "The General Irrelevance of the General Impossibility Theorem," *Quarterly Journal of Economics*, 81 (May 1967), 256–270. For a review and synthesis of this literature as well as an extended bibliography, see Charles R. Plott, "Recent Results in the Theory of Voting" (Krannert School of Industrial Administration, Paper #281, June, 1970); and William H. Riker, "Voting and the Summation of Preferences," *American Political Science Review*, 55 (December 1961), 900–911. For the general statement of the paradox and its implications see Kenneth J. Arrow, *Social Choice and Individual Values* (New York: John Wiley & Sons, 1963).

[2] For additional discussion of the relationship between the paradox of voting and spatial models see Otto A. Davis, Melvin J. Hinich, and Peter C. Ordeshook, "An Expository Development of a Mathematical Model of the Electoral Process," *American Political Science Review*, 64 (June 1970), 426–448, and Martin Shubik, "A Two-Party System, General Equilibrium and the Voter's Paradox," *Zeitschrift für Nationalökonomie*, 28 (1968), 341–354.

equilibria usually require symmetric utility functions, symmetric distributions of preferences, and citizens who weight the issues in an identical fashion, who employ the same calculus of voting, and who choose to vote for a candidate or to abstain without partisan bias.

We present here a new class of sufficient conditions for the existence of a majority-rule equilibrium and an electoral equilibrium. We review briefly in the next section the sufficient conditions that Duncan Black, Charles R. Plott, and Gordon Tullock formulate. We discuss in Section II a fundamental theorem of game theory that demonstrates the logical character of our analysis. We present in Section III conditions sufficient to guarantee the existence of an equilibrium if candidates maximize either plurality or votes.[3] While the sufficient conditions we review in Section I are constraints on the preferences citizens hold and assume that all citizens vote and choose deterministically, the conditions we offer constrain citizen-decision rules and assume that citizens choose probabilistically and can abstain.[4] We review in Section IV the assumptions of the basic multidimensional model of spatial competition that we eliminate with the new conditions we present. These now expendable assumptions are: (1) the electorate's distribution of preferences, $f(\mathbf{x})$, is symmetric; (2) all citizens assign the same relative saliency to an issue;

[3] For a rigorous distinction between vote-maximizing and plurality-maximizing candidates, see M. J. Hinich and P. C. Ordeshook, "Plurality Maximization vs. Vote Maximization: A Spatial Analysis with Variable Participation." *American Political Science Review*, 64 (September, 1970), 772–791.

[4] Our use of probability should not be confused with those analyses which seek to calculate the probability that a paradox occurs. See, for example, Richard G. Niemi and Herbert F. Weisberg, "A Mathematical Solution for the Probability of the Paradox of Voting," *Behavioral Science*, 13 (July 1968), 317–323; Richard G. Niemi, "Majority Decision-Making with Partial Unidimensionality," *American Political Science Review*, 63 (June 1969), 488–497; Frank Demeyer and Charles R. Plott, "The Probability of a Cyclical Majority," *Econometrica*, 38 (March 1970), 345–354; Mark B. Garman and Morton I. Kamien, "The Paradox of Voting: Probability Calculations," *Behavioral Science*, 13 (July 1968), 306–316; David Klahr, "A Computer Simulation of the Paradox of Voting," *American Political Science Review*, 60 (June 1966), 384–390. In these analyses, preferences are assigned probabilistically to people, and each person then votes for one alternative or another with certainty in accordance with his assigned preference ordering. Moreover, these analyses all assume that citizens vote, that is, that no one can abstain. We assume, however, that citizens' utility functions are known with certainty, but that, aggregating across the population, these utility functions define only probabilistic choices and that one admissible choice is to abstain.

(3) all citizens employ the same calculus of voting, that is, they all abstain for equivalent reasons; (4) citizen choices are unbiased by partisan attachments or candidate personality. We present in Section v several examples of situations that satisfy our conditions. Finally, we present in Section vi a numerical procedure for calculating the location of electoral equilibria as well as two conditions under which the mean preference of the electorate on each issue is the equilibrium. Because rigorous proofs of our conclusions are mathematically complex, we place them as well as the tedious technical details of our analysis, in an Appendix.

I

Two Sufficient Conditions for Majority-Rule Equilibrium

The most widely-cited condition for guaranteeing that a majority-rule equilibrium exists is Black's condition of single peakedness.[5] This condition is discussed extensively in the literature of political theory and we need not review it here. We observe, simply, that single peakedness is a constraint on citizens' preferences: that is, citizens cannot hold certain preference orderings simultaneously. If, for example, one of three citizens has the preference ordering $xPyPz$ (read: x is preferred to y and y is preferred to z) and if the second citizen orders the alternatives $zPxPy$, then a third citizen cannot order the alternatives $yPzPx$.

We could employ Black's condition as a base upon which to construct a spatial model; but it limits the generality of these models. Specifically, such a base limits us to a unidimensional conceptualization of election competition. While a single dimension or issue may characterize many elections, Stokes and Converse argue effectively that a truly realistic spatial model must be multidimensional.[6] Stated differently, we must

[5] *Theory of Committees.*
[6] Phillip E. Converse, "The Nature of Belief Systems in Mass Publics," in David E. Apter, ed., *Ideology and Discontent* (New York: Free Press, 1964), 206–261; "The Problem of Party Distances in Models of Voting Change," in M. Kent Jennings and L. Harmon Ziegler, eds., *The Electoral Process* (Englewood Cliffs, N. J.: Prentice-Hall, 1966), 175–207; and Donald E. Stokes, "Spatial Models of Party Competition," *American Political Science Review*, 57 (June 1963), 368–377.

allow the possibility that we cannot order the citizens' policy preferences
so that they are all single peaked.

But single peakedness is a sufficient though not a necessary condition
for equilibrium. Hence, Black's analysis does not preclude the possibility
of establishing weaker sufficient conditions; Tullock suggests such a
multidimensional condition and Plott formulates it explicitly.[7]
Reviewing a somewhat less general form of this condition without
attempting to do justice to its formal elegance, suppose (1) that the
election involves n issues; (2) that we can identify each citizen's most
preferred position on each issue; and (3) that citizens do not all agree
as to what policies are most desirable. Then, if the electorate's pre-
ferences are distributed symmetrically on each and every issue, and,
stated imprecisely, if all citizens weight the relative saliency of issues
identically, the mean preference is a majority-rule equilibrium.

The importance of this condition is not only that it extends Black's
analysis, but also that it is the logical complement to many of the
theorems of spatial theory. Thus, theorems that establish the existence
of a multidimensional electoral equilibrium assume, typically, symmetric
preference distributions and common patterns of issue saliency.

We generalize several of these theorems by departing from the
analyses of Black et al. in three ways. First, observe that the conditions
we review in this section assume that citizens choose deterministically
and that all citizens vote, except perhaps those who are absolutely
indifferent to the motions presented. That is, if θ and ψ are the election
alternatives, and if the citizen prefers θ to ψ, then he votes for the
candidate who advocates θ with a probability of one. We assume,
however, that citizens choose *probabilistically* and that they can
abstain. Therefore, instead of assuming that if a citizen prefers θ to ψ
he must vote for the candidate who advocates θ, we assume, for
example, that the citizen is more likely to vote for the candidate who
advocates θ than for the candidate who advocates ψ, and that he might
abstain.[8]

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[7] Black, *Theory of Committees*; Tullock, "Impossibility Theorem"; and Plott,
"Equilibrium."

[8] A probabilistic model of individual choice in the context of rational decision-
making is not original with this essay—although the application of such a model to
spatial theory and the paradox of voting is. For a discussion of alternative inter-
pretations of probabilistic choice see the appendix to Niemi and Weisberg, "The
Effects of Group Size and Collective Decision-Making," in Niemi and Weisberg,

Second, we observe that, like single peakedness, the multidimensional condition consists fundamentally of restrictions on the preferences that citizens can hold. By requiring that the distribution of preferences is symmetric, for example, constraints such as "if the first $m-1$ citizens' preferences are distributed . . . , then, to insure symmetry, the mth citizen's preference must be . . . " are imposed. Rather than constrain the distribution of preferences, however, we constrain the relationship between the likelihood of the citizen's choices and θ and ψ. Thus, we might assume that the probability that the citizen votes for the candidate who advocates θ increases as his utility for θ increases.

The final difference concerns the location of equilibria. Plott et al. demonstrate with the symmetry assumptions not only that an equilibrium exists but also that it is the mean citizen preference. If we add these assumptions to our analysis, the social choice remains the mean citizen preference. We present in Section vi however, a theorem that establishes the mean as the equilibrium though it does not require that the citizens' preferences are distributed symmetrically. Moreover, we offer in that section a numerical procedure for calculating the location of equilibria in general.

II

A General Theorem from the Theory of Games

Before we describe our sufficiency conditions, we demonstrate the logic of our formal analysis by noting the relevance of two-person game theory to spatial theory. Assume that two candidates, one and two, compete, and let θ and ψ denote their respective spatial strategies. Since the return to a candidate depends on not only the strategy he employs but also the strategy his opponent employs, the election is a two-person game. If we assume that the payoffs to these candidates are their expected pluralities, denoted by $\varphi_1(\theta, \psi)$ and $\varphi_2(\theta, \psi)$, the game is zero-sum because $\varphi_1(\theta, \psi) = -\varphi_2(\theta, \psi)$. And if we assume, as

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eds., *Probability Models of Collective Decision-Making* (Columbus, Ohio: Charles E. Merrill, 1972), 146–148. For a review of the tradition of this model in psychology— the discipline in which it is most extensively employed—see, for example, Clyde Coombs, Robyn Dawes and Amos Tversky, *Mathematical Psychology* (Englewood Cliffs, N. J.: Prentice-Hall, 1970), chap. 5.

is generally the case in spatial theory, that θ and ψ are continuous variables (for example, are measured on some segment of the real line), the game is an infinite game. Thus, we can conceptualize spatial competition between two candidates whose payoffs are expected pluralities as a two-person, zero-sum, infinite game.

Pure minimax ("best of the worst") strategies for such games are strategies, say θ^* and ψ^*, that satisfy the following conditions:[9]

$$
\begin{array}{ll}
\varphi_1(\theta^*, \psi) \geqslant \varphi_1(\theta^*, \psi^*) \geqslant \varphi_1(\theta, \psi^*) & \text{if } \theta \neq \theta^* \\
\varphi_2(\theta, \psi^*) \geqslant \varphi_2(\theta^*, \psi^*) \geqslant \varphi_2(\theta^*, \psi) & \text{if } \psi \neq \psi^*
\end{array}
\tag{1}
$$

That is, (θ^*, ψ^*) is an equilibrium strategy pair if neither player has an incentive unilaterally to alter his strategy once the two players arrive at this pair. Since $\varphi_1(\theta^*, \psi^*) \geqslant \varphi_1(\theta, \psi^*)$, player one might decrease his payoff by shifting from θ^* to some other strategy θ. Such a shift

[9] We are concerned in this essay only with pure strategy solutions since mixed strategies make little or no sense in a theory of the electoral process. To see this we must clarify the distinction between mixed strategies and strategies involving risk. First, a pure stategy, θ, in our model is a specific position on each of n issues. A risky strategy is a probability distribution over the set of pure strategies which the candidate, perhaps implicitly, tells the voter he will use in choosing a policy *after* the balloting. Mixed strategies, however, concern only the mechanism by which some pure strategy is selected. The decision to adopt a mixed strategy means that the candidate selects a pure strategy probabilistically *before* the balloting and presents some pure strategy to the electorate for approval. Thus, if the candidate chooses to use a mixed strategy, he eventually presents the electorate with a pure strategy. It does not seem reasonable now to suppose that candidates actually adopt mixed strategies in a campaign: the inclusion of mixed strategies as an alternative means—to apply the theory of games properly—that candidates reveal their pure strategies simultaneously and that a candidate cannot alter his strategy once it is revealed (for example, the payoffs are realized immediately). Obviously, however, payoffs are not realized immediately in elections, and candidates attempt to change their strategies if they are inferior to those of their opponents. We cannot imagine a candidate adopting a mixed strategy which results in the selection of a pure strategy that is inferior to his opponent's strategy without his attempting to alter his position. Since such inferior choices are entirely possible in the theory of games, we do not consider mixed solutions to electoral competition. Also, we do not consider the second general type of strategy that candidates might adopt—risky strategies—because, given our assumption that utility functions are concave, it is relatively easy to show that risky strategies cannot constitute an equilibrium. Consequently, our search for equilibria is constrained to pure strategies. For an analysis of risky strategies see Kenneth Shepsle, "The Strategy of Ambiguity: Uncertainty in Electoral Competition," *American Political Science Review*, 66 (June 1972), 555–568.

provides no hope of a gain. Similarly, since $\varphi_2(\theta^*, \psi^*) \geqslant \varphi_2(\theta^*, \psi)$, player two has no incentive to shift unilaterally from ψ^*.

We can specify two constraints on two-person, zero-sum, infinite games that guarantee the existence of a unique pure strategy pair (θ^*, ψ^*).[10] These constraints are:

   (i) φ_1 and φ_2 are everywhere continuous in θ and ψ,

   (ii) φ_1 is concave in θ and convex in ψ; φ_2 is concave in ψ and convex in θ.[11]

(Parenthetically, to avoid confusion, we note the distinction between the analysis of finite and infinite zero-sum games. Finite games are games in which the players have a finite number of pure strategies. We can represent such games as a matrix in which the rows denote the pure strategies of player one, the columns denote the pure strategies of player two, and the entries of the matrix denote the payoffs to one or both players. The classic result of the theory of two-person games—the minimax theorem—is that every two-person, zero-sum, *finite* game possesses at least one pair of pure or mixed strategies that satisfies (1).

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[10] For a more precise statement of the theorem, see R. Duncan Luce and Howard Raiffa, *Games and Decisions* (New York: John Wiley & Sons, 1964), 452, and Guillermo Owen, *Game Theory* (Philadelphia: W. B. Saunders, 1968), 78–81. While several of our theorems follow directly from the results reviewed in these references, our theorems concerning constrained strategies as well as the numerical procedure we offer later for calculating equilibria follow from the analysis of J. B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," *Econometrica*, 33 (July 1965), 520–534. The proofs in our Appendix, then, are directed primarily towards satisfying Rosen's conditions.

[11] By continuous we mean roughly that a graph of φ_i against θ or ψ contains no instantaneous changes in value. By concave in θ we mean that

$$\varphi_1(\xi\theta' + (1 - \xi)\theta'', \psi) \geqslant \xi\varphi_1(\theta', \psi) + (1 - \xi)\varphi_1(\theta'', \psi)$$

for all θ', θ'' and ψ, where $0 < \xi < 1$. By convex in ψ we mean that,

$$\varphi_1(\theta, \xi\psi' + (1 - \xi)\psi'') \leqslant \xi\varphi_1(\theta, \psi') + (1 - \xi)\varphi_1(\theta, \psi'')$$

for all θ, ψ', and ψ''. (For our analysis, strict inequality holds for some values of θ.) Geometrically, a curve is concave if, for any two points on it, a straight line between them never lies above the curve. And a curve is convex if, for any two points on it, a straight line between them never lies below the curve. Hence, if we plot φ_1 against θ, the line connecting θ' and θ'' lies below the graph of φ_1 in the interval $[\theta', \theta'']$; and if we plot φ_1 against ψ, the line connecting ψ' and ψ'' lies above the graph of φ_1 in the interval $[\psi', \psi'']$.

This result does not hold generally, however, for two-person, zero-sum, *infinite* games. Nevertheless, if we impose (i), at least a mixed-strategy equilibrium exists, and if we also impose (ii), a unique pure-strategy equilibrium exists.)

As an important caveat to this result, suppose that the game is symmetric. That is, suppose (1) that both players choose from identical sets of strategies, and (2) that if the players exchange strategies, they exchange payoffs, $\varphi_1(\theta, \psi) = \varphi_2(\psi, \theta)$. Hence for such games, both players possess the same equilibrium strategy ($\theta^* = \psi^*$), and in equilibrium, they both receive the payoff of zero, that is, $\varphi_1(\theta^*, \psi^*) = \varphi_2(\theta^*, \psi^*) = 0$.

The relevance of these results to spatial theory is evident if we retrace the correspondence between election competition and game theory. The existence of pure minimax strategies for the game theoretic model of election competition means that each candidate can find a pure strategy that guarantees him some minimal payoff and from which he has no incentive to move if his opponent also adopts a minimax strategy. Furthermore, if the election is symmetric—a case that we consider in greater detail later—both candidates can secure the expectation of at least a tie, that is, $\varphi_1(\theta^*, \psi^*) = 0$; if both candidates adopt their equilibrium strategies, they converge, that is, $\theta^* = \psi^*$.

The relevance of these results to the paradox of voting is evident if we let $\varphi_1(\theta^*, \theta)$ denote the number of people who vote for θ^* minus the number of people who vote for θ. Assuming that (i) and (ii) are satisfied and that the game is symmetric, we know now that $\varphi_1(\theta^*, \theta) > 0$ if $\theta^* \neq \theta$, θ^* receives more votes than θ, and that θ^* is the only motion that can achieve majority approval against every other admissible motion.

This correspondence among game theory, spatial analysis, and the paradox of voting defines our objective: to establish assumptions about the electorate's calculus of voting which yield payoff functions that satisfy conditions analogous to (i) and (ii) and that permit us to interpret these conditions substantively.

III

A sufficient Condition for Equilibrium

Our sufficient condition for the existence of election equilibria consists of two classes of assumptions. The first class concerns the

utility that citizens associate with the candidates' strategies; the second class concerns the citizens' decision calculus. First, let $X = (X_1, X_2, \ldots, X_n)$ denote a position on each of n issues and let $U(X/x)$ denote the utility that a citizen associates with X, given that he most prefers $x = (x_1, x_2, \ldots, x_n)$.[12] Clearly, then, the utility function $U(X/x)$ is maximized at $X = x$. And if we let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ denote candidate one's position on each of the n issues, the function $U(\theta/x)$ is the utility the citizen associates with the election of candidate one. Our first assumption concerns the form of the function U.

ASSUMPTION A1: $U(X/x)$ *is continuous and concave in* X.[13]

Stated in spatial terms, assumption A1 implies that the utility a citizen associates with a candidate decreases at a constant or an increasing rate as that candidate moves his strategy, θ, away from the citizen's most preferred position, x. Thus, with A1 we do not permit the citizen's utility function to "level off," so that he becomes indifferent between, say, two "conservative" policies that are far from his ideal. We note by way of example that A1 is satisfied if we measure utility loss simply by spatial distance.

Constraint A1, however, is not sufficient to guarantee that the candidates' payoff functions, φ_1 and φ_2, possess the continuity, convexity, and concavity properties we outline in the previous section. To generate such payoff functions we must relate θ and ψ to a citizen's choice with a second class of assumptions. That is, a citizen may choose: (1) to vote for candidate one; (2) to vote for candidate two; or (3) to abstain. We must define the relationship between the candidates' strategies and the citizen's choice of one of these three actions.

Recall that we establish with A1 the relationship between the can-

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[12] Note that we write U as conditional on x. We adopt this notation because we do not require the assumption that U has the same functional form for any two or more citizens. Thus, even if two people prefer the same policies, we do not assume that the forms of their utility functions are similar.

[13] Thus, from the definition presented in fn. 11,

$$U(\xi X' + (1 - \xi) X''/x) \geqslant \xi U(X'/x) + (1 - \xi) U(X''/x)$$

For our theorems we require that this inequality be strict for some values of X' and X''. Also, we assume that the space of all possible strategies is compact. That is, x_i and X_i vary only in some finite continuous interval such as $[0, 1]$. Substantively, this eliminates strategies and preferences at $+$ or $-$ infinity.

didates' strategies, θ and ψ, and the citizen's utilities, $U(\theta/\mathbf{x})$ and $U(\psi/\mathbf{x})$. Now we must define the relationship between the citizen's utilities and the probability that he chooses one of the three actions. This establishes, somewhat circuitously, the relationship between the strategies θ and ψ, and the citizen's choice.

Since the number of assumptions that we might impose to define this relationship is infinite, we limit our alternatives by appealing to some behavioral heuristics. We wish to know what logical relationships between a citizen's utility for θ and for ψ and a citizen's choice appear reasonable. We answer this question thus:

(a) The probability that a citizen votes for candidate i increases or at worst remains a constant as the utility he associates with candidate i's position increases.

(b) The probability that a citizen votes for candidate i decreases or at best remains a constant as the utility he associates with candidate j's position increases, $i \neq j$.[14]

To formulate heuristics (a) and (b) rigorously, we denote the probability that a citizen votes for candidate one by $p_1(U(\theta/\mathbf{x}), U(\psi/\mathbf{x}))$, and we denote the probability that he votes for candidate two by $p_2(U(\theta/\mathbf{x}), U(\psi/\mathbf{x}))$. Two formal interpretations are consistent with heuristic (a). First,

ASSUMPTION A2: p_1 *increases as* $U(\theta/\mathbf{x})$ *increases, and* p_2 *increases as* $U(\psi/\mathbf{x})$ *increases.*

Second, an alternative to A2 that also is reasonable and that captures the intuitive idea behind (a) is,

ASSUMPTION A2': p_1 *increases as* $U(\theta/\mathbf{x})$ *increases if* $U(\theta/\mathbf{x}) \geqslant U(\psi/\mathbf{x})$; *otherwise* $p_1 = \gamma$, $0 \leqslant \gamma < 1$, *and* p_2 *increases as* $U(\psi/\mathbf{x})$ *increases if* $U(\psi/\mathbf{x}) \geqslant U(\theta/\mathbf{x})$; *otherwise* $p_2 = \gamma$.

Note the difference between A2 and A2'. In A2, p_1 increases for all

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[14] For an empirical analysis that is explicitly designed to test these two hypotheses and which strongly supports the first but only weakly supports the second, see Howard Rosenthal, "Electoral Participation in the French Fifth Republic: The FY Vote," *American Political Science Review*, in press.

values of θ as $U(\theta/x)$ increases, which is to say that the citizen's probability of voting for candidate one, p_1 might never equal zero and is subject to change for all values of θ. We assume, then, that the citizen may have a nonzero probability of voting for candidate one, even though he prefers candidates two's spatial strategy, ψ, to candidate one's spatial strategy, θ (we can require, however, that the citizen be more likely to vote for candidate two than candidate one if he prefers ψ to θ). Hence, we can interpret assumption A2 as permitting decisions that appear irrational: a citizen might choose to vote for the candidate who proposes the less preferred alternative. In lieu of relegating such acts to the realm of the irrational, however, observe that it is difficult, and perhaps impractical to represent many of the criteria citizens use to evaluate candidates—such as the candidates' personalities—in spatial terms. Instead of attempting to conceptualize and to measure such criteria so that we can represent them as spatial dimensions, assumption A2 allows us to leave considerations like the candidates' personalities as variables that are exogenous to a spatial analysis. Thus, while the citizen prefers ψ to θ for the identified spatial dimensions, he may vote for candidate one because of the positive weight he assigns to the candidate's personality. We require only that personality act as a random variable so that the citizen votes for candidate one probabilistically.

Consider now assumption A2'. Again we can permit choices that appear irrational by setting $\gamma > 0$. Here, though, γ is a constant. Observe, however, that we can eliminate "irrationality" entirely with A2' if we set $\gamma = 0$. If $\gamma = 0$, the citizen votes for candidate one only if he prefers θ to ψ.

Turning now to heuristic (b), we interpret this heuristic formally with the following assumption.

ASSUMPTION A3: p_1 *decreases or remains unchanged as* $U(\psi/x)$ *increases, and* p_2 *decreases or remains unchanged as* $U(\theta/x)$ *increases.*

To achieve the desired form for the candidates' payoff functions, we add the following assumption that constrains further the relationships among p_1, p_2, and $U(\theta/x)$, $U(\psi/x)$ in assumptions A2, A2' and A3:

ASSUMPTION A4: p_1 *is concave in* $U(\theta/x)$ *and convex in* $U(\psi/x)$, *and* p_2 *is concave in* $U(\psi/x)$ *and convex in* $U(\theta/x)$.

We demonstrate in subsequent sections that two relationships between the citizen's over-all probability of voting, $p_1 + p_2$, and his utilities for each candidate's position, $U(\theta/x)$ and $U(\psi/x)$, that other spatial models describe as alienation and indifference, are consistent with our assumption.[15] Specifically, we show that assumptions A2 through A4 permit us to use the alienation and the indifference hypotheses as descriptions of a citizen's calculus of voting either separately or simultaneously. But before we examine the substantive implications of these assumptions in greater detail—especially A4, which we cannot interpret readily without recourse to examples—we present the theorems that establish the existence of equilibrium strategies.

Our first theorem pertains to candidates that maximize their expected plurality:

> THEOREM 1: *If each candidate maximizes his expected plurality, and if either assumption set A1, A2, A3, and A4 or assumption set A1, A2' A3, and A4 holds for all citizens, then a unique pure strategy equilibrium exists.*

That is, if we let $\varphi_i(\theta, \psi)$ denote candidate i's expected plurality, there exists a strategy pair (θ^*, ψ^*) such that expression (1)—the minimax condition—is satisfied.

From the perspectives of spatial theory and the voting paradox, however, a most important corollary to Theorem 1 is,

> COROLLARY 1.1: *If in addition to the conditions of Theorem 1, p_1 and p_2 are of identical functional form for each citizen, then $\theta^* = \psi^*$.*

Hence, if we denote the common value of θ^* and ψ^* by α (i.e., $\theta^* = \psi^* = \alpha$), then from (1),

[15] For explicit spatial formulations of indifference and alienation, see Davis, Hinich, and Ordeshook, "Expository Development," on alienation, Hinich and Ordeshook, "Abstentions and Equilibrium in the Electoral Process," *Public Choice*, 7 (Fall 1969), 81–106; on indifference, Ordeshook, "Some Extensions to a Mathematical Model of the Electoral Process," *Midwest Journal of Political Science*, 14 (February 1970), 43–70; and Richard McKelvey, "Some Extensions and Modifications of a Spatial Model of Party Competition" (unpublished Ph.D. dissertation, University of Rochester, 1972). These two hypotheses are also described by Gerald Garvey, "The Theory of Party Equilibrium," *American Political Science Review*, 60 (March 1966), 29–38.

$$\varphi_1(\alpha, \alpha) = 0 \geqslant \varphi_1(\theta, \alpha) \qquad \text{if} \quad \theta \neq \alpha$$

$$\varphi_2(\alpha, \alpha) = 0 \geqslant \varphi_2(\alpha, \psi) \qquad \text{if} \quad \psi \neq \alpha$$

That is, if both candidates are at α and if either candidate shifts unilaterally from α, his expected plurality does not increase. It may decrease and become negative.

Theorem 1 states a sufficient condition for the existence of a unique, pure-strategy equilibrium in a two-person, zero-sum, infinite game: the conditions of Corollary 1.1 guarantee that the two players share the same equilibrium-strategy solution. With respect to the paradox of voting, then, this corollary yields a sufficient condition for the existence, in a multidimensional issue space, of a unique alternative that defeats all other alternatives in a majority vote. Interpreting this result for spatial theory, the assumption that p_1 and p_2 are of identical functional form means that citizens evaluate the candidates only on the basis of the candidates' spatial positions, that is, that the citizens are not biased in favor of one candidate or the other. Thus, if the conditions of Theorem 1 hold and if citizens are unbiased, Corollary 1.1 states that a unique electoral equilibrium exists such that the candidates adopt identical strategies, that is, they converge.

Our second existence theorem pertains to candidates that maximize their expected vote,

THEOREM 2: *If each candidate maximizes his expected vote and if assumptions A1, A2, A3, and A4 hold for all citizens, then a unique pure-strategy equilibrium exists.*[16]

Additionally, we prove a corollary to Theorem 2 that parallels Corollary 1.1.

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[16] In addition to assumptions A1 through A4 we also require for uniqueness for vote maximization that,

$$\frac{\partial^2 p_1}{[\partial U(\theta/x)]^2} \cdot \frac{\partial^2 p_2}{[\partial U(\psi/x)]^2} - \left[\frac{\partial^2 p_1}{\partial U(\theta/x) \, \partial U(\psi/x)} + \frac{\partial^2 p_2}{\partial U(\psi/x) \, \partial U(\theta/x)} \right]^2 \geqslant 0$$

Note that this condition is satisfied if the cross-partials of p_i are zero; that is, if the *rate of change* in a citizen's probability of voting for one candidate is independent of the other candidate's strategy. Any further attempt to interpret this condition substantively is probably hopeless. Hence, of those functions that satisfy A1 through A4, we consider only examples that satisfy this condition, except for example (5).

COROLLARY 2.1: *If, in addition to the conditions of Theorem 2,* p_1 *and* p_2 *are of identical functional form for each citizen, then* $\theta^* = \psi^*$.

Theorems 1 and 2 state that our assumptions are sufficient for the existence of an electoral equilibrium with either plurality or vote maximization while their corollaries establish sufficient conditions for the candidates to converge to identical strategies. Observe, however, that in both theorems we permit the candidates to adopt any strategy. This is the assumption that analyses of electoral equilibria and the paradox of voting generally employ. Downs, however, suggests that the candidates are frequently constrained so that they cannot cross each other on a dimension.[17] If candidate one is identified by the electorate as being prolabor, for example, candidate two might be unable to convince the electorate that he is more prolabor than his opponent. Or, for another example of a constraint, a candidate's position on one issue may constrain his freedom of choice on a different issue. Thus, if a candidate has a strong position on law and order, a citizen might not perceive him to hold a liberal position on civil rights.

To incorporate such possibilities into our analysis, suppose that each candidate's strategy choice is constrained in $R \geqslant 1$ ways. Specifically, assume that,

(2)　　　$G_r(\theta, \psi) \geqslant 0$　　for　$r = 1, 2, \ldots, R$, where each G_r is concave in θ and ψ.[18]

To illustrate (2), suppose that a single constraint is imposed, namely that the candidates cannot cross each other on issue i. For example, suppose that θ_i and ψ_i must satisfy $\theta_i \leqslant \psi_i$. Thus, $G_1(\theta, \psi) = \psi_i - \theta_i \geqslant 0$. Alternatively, if the candidates cannot cross on *any* issue, we can let $G_1(\theta, \psi) = \psi - \theta \geqslant 0$. As a second illustration, suppose

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[17] Anthony Downs, *An Economic Theory of Democracy* (New York: Harper and Row, 1957), chap. 8 and 9.

[18] That is,

$$G_r(\xi\theta' + (1 - \xi)\,\theta'', \psi) \geqslant \xi G_r(\theta', \psi) + (1 - \xi)\,G_r(\theta'', \psi)$$

and

$$G_r(\theta, \xi\psi' + (1 - \xi)\,\psi'') \geqslant \xi G_r(\theta, \psi') + (1 - \xi)\,G_r(\theta, \psi'')$$

To show that our examples satisfy these conditions, consider $\psi - \theta \geqslant 0$. First, if we expand

that the only constraint imposed is that a candidate's position on one issue constrains the positions he can take on some other issue. Specifically, let $\theta_i - \theta_j \geqslant 0$. Hence, if candidate one adopts a liberal position on issue i (denoted by negative values of θ_i), he must adopt at least an equally liberal position on issue j (denoted by negative values of θ_j). Thus, $G_1(\theta, \psi) = \theta_i - \theta_j \geqslant 0$. Finally, since we can have any number of constraints, we are able to impose both of our examples of constraints simultaneously; that is, we can let $R = 2$, $G_1(\theta, \psi) = \psi - \theta \geqslant 0$ and $G_2(\theta, \psi) = \theta_i - \theta_j \geqslant 0$.

We propose, then, to use (2) to represent formally the constraints on the candidates' strategies that usually characterize elections. The first theorem pertains to plurality maximizing candidates:

THEOREM 3: *If the conditions of Theorem 1 hold and if (2), a unique pure strategy equilibrium exists.*

Similarly, for vote maximizing candidates,

THEOREM 4: *If the conditions of Theorem 2 hold and if (2), a unique pure strategy equilibrium exists.*

Thus, even with the impositions of constraints, an equilibrium exists, provided these constraints satisfy (2).

Finally, if we assume that the conditions of Corollaries 1.1 and 2.1 hold and assume that (2) does not prohibit the candidates from converging, (that is, $G_r(x, x) \geqslant 0$ for all x), then for plurality-maximizing candidates:

COROLLARY 3.1: *If, in addition to the conditions of Theorem 3 and Corollary 1.1, $G_r(x, x) \geqslant 0$ for all x and r, then $\theta^* = \psi^*$.*

and for vote-maximizing candidates,

$$\psi - [\xi\theta' - (1 - \xi)\theta''] = \xi[\psi - \theta'] + (1 - \xi)[\psi - \theta'']$$

we find that strict equality holds. Similarly, strict equality holds for

$$[\xi\psi' - (1 - \xi)\psi''] - \theta \geqslant \xi[\psi' - \theta] + (1 - \xi)[\psi'' - \theta]$$

Hence, $\psi - \theta \geqslant 0$ is an admissible constraint. Similarly, we can show that $\theta_i - \theta_j \geqslant 0$ is admissible.

COROLLARY 4.1: *If, in addition to the conditions of Theorem* 4 *and Corollary* 2.1, $G_r(\mathbf{x}, \mathbf{x}) \geqslant 0$ *for all* $\mathbf{x}$ *and* r, *then* $\theta^* = \psi^*$.

Thus, if the candidates converge without any constraints on their strategies, and if we impose constraints that satisfy (2) but that do not prohibit convergence, then the candidates converge.

IV

Some Unnecessary Assumptions

The conclusion that a unique equilibrium exists may not surprise many readers. After all, if we make enough assumptions, we can prove almost anything. In this section, therefore, we consider the assumptions that we do not use in the proofs of Theorems 1 through 4 and their corollaries, but that previous spatial analyses employ.

First, recall from our discussion in Section I that one of the conditions for the existence of an equilibrium which Plott and Tullock present is: the electorate's preference density, $f(\mathbf{x})$, is symmetric. Similarly, the results reported in "An Expository Development of a Mathematical Model of the Electoral Process" pertain predominantly to symmetric densities and frequently to only symmetric unimodal densities. But we do not impose such restrictions on $f(\mathbf{x})$, which is to say that $f(\mathbf{x})$ *can be a unimodal, a bimodal, or a multimodal density as well as a symmetric or a nonsymmetric density.*[19]

A second unnecessary assumption concerns the form of a citizen's utility function. In other multidimensional spatial models, for example, a citizen's utility function is

$$U(\theta/\mathbf{x}) = \lambda - \beta \sum_{i=1}^{n} (x_i - \theta_i)^2,$$

[19] However, a constraint of a different sort must be imposed. Since p_1 is concave in $U(\theta/x)$ and convex in $U(\psi/x)$, the mathematical possibility exists that p_1 or $p_1 + p_2$ is less than 0 or greater than 1. Because such probability numbers make no sense, we must constrain either $f(x)$ or U. This constraint can be derived from expression (13A) in our Appendix and expressed rigorously as a function of the variance of $f(x)$ and the first and second derivatives of U. Briefly, this constraint can be interpreted to mean that the variance of $f(x)$ or the relative saliency of an issue cannot be "too great." In lieu of presenting the complex mathematical form

where λ and β are positive constants.[20] To assume, however, that this expression characterizes utility functions requires that the utility function $U(\theta/x)$ be symmetric. That is, the loss a citizen incurs if a candidate shifts one unit to the left of his ideal preference, x, equals the loss he incurs if the candidate shifts one unit to the right of x. According to previous models, then, a citizen cannot most prefer a moderate policy while his remaining preferences are biased either to the right or to the left. *Assumption A1 imposes concavity—an assumption that our above example satisfies—but it does not impose symmetry.*

A far more important series of assumptions that we do not impose concerns the electorate's conformity (1) to a pattern of issue saliency, and (2) to a calculus of voting. Nearly all of the theorems pertaining to multidimensional spatial competition assume that all citizens weight the relative saliency of the issues identically. Hence, one subset of the electorate cannot care primarily about civil rights while another subset cares most about farm subsidies. Theorems 1 through 4 and their corollaries, however, do not require any assumptions about issue-saliency patterns that we might find in any electorate. Thus, some citizens can be concerned primarily with farm subsidies while some other subset of the electorate might study only the candidates' positions on civil rights, unemployment, or war, and yet other groups are concerned with several of these issues simultaneously. Stated differently, *while each citizen's utility function must be concave, the functional form of a citizen's utility function can be entirely different from that of any other citizen.*

Second, recall that the conditions for equilibrium that we review in Section I assume that all citizens employ identical decision rules: they all vote. Similarly, all citizens must still employ an identical calculus of voting in those theorems of spatial analysis that permit citizens to abstain because of alienation or because of indifference. Generally, indifference cannot explain abstention for one subset of the electorate while alienation explains abstention for some other subset. Furthermore, if two citizens share the same ideal preference, they must vote with the same probability. Our theorems do not suppose that the citizens all share the same calculus of voting. Manifestly p_i *can be an*

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of this constraint, however, we simply note its existence and the necessity of making sure that all the probabilities lie in the interval [0, 1].

[20] For a discussion of the quadratic metric in the context of spatial theory see Davis, Hinich, and Ordeshook, "Expository Development," 429–436.

entirely different function for each citizen. Some citizens can abstain because of indifference, others because of alienation, and still others because of some combination of indifference and alienation. And if two citizens both abstain for identical reasons, one citizen's choice can be more sensitive to the candidates' strategies than the other citizen's choice. Hence, if two citizens share the same ideal preference, we do not require that they vote for a candidate with the same probability.

Finally, recall that in spatial theory the candidates' labels are inter-changeable. That is, if initially candidates one and two adopt θ and ψ respectively, and if they switch strategies so that candidate one is now at ψ, candidate one's plurality equals the plurality candidate two received when he was at ψ. This feature of spatial theory can be inter-preted to mean that citizens are cognizant only of the candidates' spatial strategies: they vote for candidate one if he is at ψ with the same probability that they vote for candidate two if he is at ψ. *But in Theorems 1 through 4, it is unnecessary to assume that p_1 is the same function as p_2*. Hence, a citizen can be more sensitive to variations in candidate two's strategy. Or, a citizen can vote for candidate one with a low probability even though θ is close to x, because this candidate is a Republican or because he finds the candidate's public image distasteful. But if candidate two is at θ, the citizen votes for two with a high pro-bability because two is a Democrat or because the candidate's public image attracts the citizen. Finally, and in accordance with our previous discussion, some citizens can be biased in favor of candidate one while others are biased in favor of candidate two.[21] Note that this permits us to have a situation in which the entire electorate, or at least some majority of the electorate, is biased in favor of candidate one so that, although a pure strategy equilibrium solution exists, candidate one can secure the expectation of a positive plurality.[22]

Our freedom to assume that citizens in our model are biased leads

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[21] We must emphasize, however, that our formulation of bias is not the same thing as cognitive balance. That is, we continue to assume that all citizens behave as if they make identical estimates of θ and ψ. Our notion of bias is that if $\theta = \psi$, for example, the probability that a citizen votes for candidate one can exceed his probability of voting for candidate two.

[22] To see this, recall that the value to a player of a zero-sum game is not necessarily zero. For example, in the zero-sum game we illustrate below (in which payoffs are to player one), the value of the game to player one is -2.

us to question the generality of some results from earlier spatial analyses. Specifically, it is typically the case in those studies that plurality-maximizing candidates converge if the electorate's preferences are distributed unimodally. These analyses assume, however, that citizens have no biases. But if the function p_1 is not equivalent to the function p_2 because of the presence of such bias, we cannot guarantee that the candidates converge. Consider, for example, an election in which: (1) the electorate's preferences are distributed unimodally; (2) citizens are concerned with one common issue; and (3) citizens with ideal preferences to the left of the mean are biased in favor of candidate one while citizens to the right of the mean are biased in favor of candidate two. There is no reason to suppose now that $\theta^* = \psi^*$. The equilibrium probably is $\theta^* < \psi^*$ because a candidate is not likely to campaign to mobilize those citizens that do not, in any circumstance, respond to his overtures. Rather, he prefers to maximize the turnout of his potential source of votes. *Thus, nonuniform distributions of bias can yield equilibrium solutions in which two plurality maximizing candidates fail to converge, although the over-all distribution of preferences in the electorate is unimodal. We can infer from this possibility that if partisan attachments bias p_1 and p_2, and if party identification correlates with preferences on an issue, an equilibrium exists, but the candidates may not converge.*

The weakening or elimination of the assumptions that we review in this section necessarily involves incurring a cost—specifically, our concavity and convexity assumptions, especially A4. Since at this point there is little empirical research that we can use to measure and evaluate these costs, in the next section we explore the properties of our assumptions with some examples of admissible formulations of a citizen's calculus.

		Player Two	
		β_1	β_2
Player One	α_1	-5	-6
	α_2	-2	-1

V

Some Examples of Admissible Formulations of a Citizen's Calculus

Perhaps one of the more surprising conclusions of our analysis is that if $p_1 \equiv p_2$—if citizens are not biased so that they act only on the basis of the candidates' spatial strategies—the candidates converge for all densities of preference. That is, an unbiased electorate guarantees that both plurality-maximizing and vote-maximizing candidates converge if $f(\mathbf{x})$ is either a unimodal or a bimodal density. We can use this result to explore the properties of our assumptions. Because previous spatial analyses show that when citizens abstain because of alienation, plurality-maximizing candidates may fail to converge if preferences are distributed bimodally, and vote-maximizing candidates may fail to converge if preferences are distributed unimodally, it is evident that our formulation of alienation differs from previous formulations.

First, by alienation we mean that a citizen's probability of voting decreases as the utility he derives from his most preferred candidate's strategy decreases. Previous spatial analyses incorporate this hypothesis in this way: (1) let g denote the probability that the citizen votes for candidate one; (2) assume that g increases as $U(\theta/\mathbf{x})$ increases if
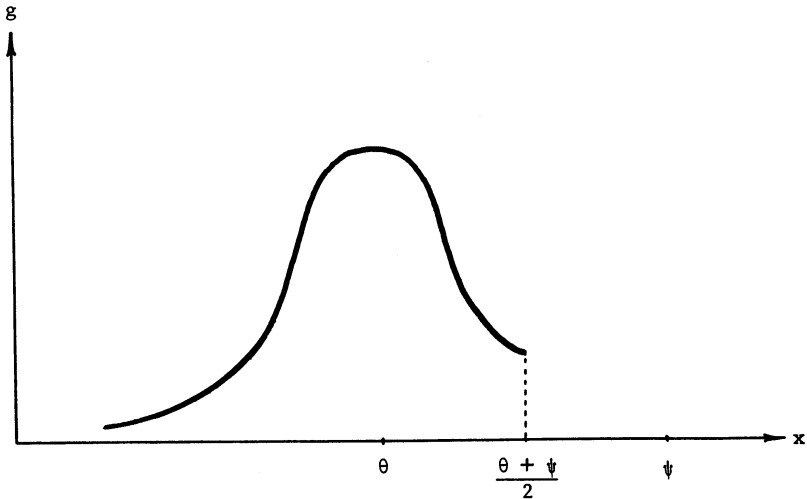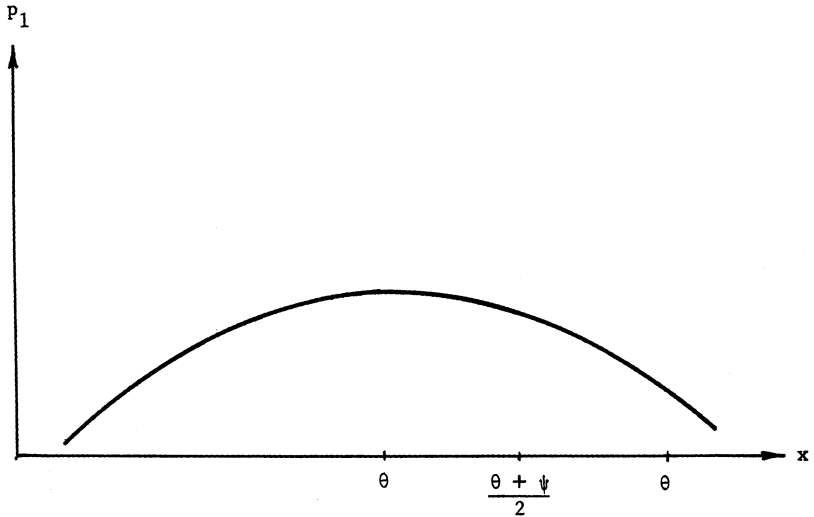
FIGURE 1

FIGURE 2

$U(\theta/x) \geqslant U(\psi/x)$; (3) assume that $g = 0$ if $U(\theta/x) < U(\psi/x)$. We illustrate in Figure 1 a typical function that satisfies these assumptions. We emphasize two properties of this function:

(1) g drops sharply from a positive value to zero at $x = (\theta + \psi)/2$: citizens are perfectly discriminating in the sense that if the citizen prefers θ to ψ, that is, if $U(\theta/x) > U(\psi/x)$, the citizen has a zero probability of voting for the candidate whose position is ψ.

(2) g is not everywhere concave: if x is far from θ, g is convex.

We could show readily that the function g fails to yield payoff functions that satisfy conditions (i) and (ii) for a unique pure-strategy equilibrium. With g the payoff functions are neither continuous nor concave. Thus, there is no reason to expect that the theorems about spatial competition that use g conform to the theorems in this essay.

To see how alienation can be formulated so that it is consistent with assumptions A1 through A4, let,

(3) $$p_1 = a[U(\theta/x)];$$
$$p_2 = a[U(\psi/x)];$$

$(da/dU \geqslant 0$ and $d^2a/dU^2 \leqslant 0$ so that a can be either a function or a positive constant).[23] We illustrate the function p_1 in Figure 2.

Note that unlike the function we illustrate in Figure 1, p_1 is greater than zero and is concave for all θ. That is, a citizen retains a positive probability of voting for the candidate whose position is θ even though his utility for ψ is greater than his utility for θ. Additionally, we see that, while a citizen's over-all probability of voting,

$$p_1 + p_2 = a[U(\theta/\mathbf{x})] + a[U(\psi/\mathbf{x})],$$

increases as $U(\theta/\mathbf{x})$ increases, it also increases as $U(\psi/\mathbf{x})$ increases. Hence, in this formulation of alienation, a citizen's probability of voting is a function of both candidates' strategies.

We cannot express a preference either for a formulation of alienation that conforms to (3) or for a formulation such as the one we illustrate in Figure 1 without some empirical evidence. Unfortunately, this evidence is not now available. It is plausible, however, that neither formulation is right or wrong, but rather that they are more or less applicable in varying situations. For example, (3) may be more applicable for analyzing elections in which many nonspatial considerations affect citizens' choices, or in which our measurement of spatial considerations is imprecise. Thus, in our abstract representation, a citizen can vote for candidate one although he prefers ψ to θ. Alternatively, a formulation such as g seems more appropriate for elections in which measurement is precise, in which only known spatial considerations are relevant, or in which the known spatial considerations are very salient. If these conditions are satisfied, it seems more reasonable to assume that citizens vote only for candidates whose positions they prefer.

But we can answer the question we set out to answer; we can identify the logic of convergence if $f(\mathbf{x})$ is a bimodal density and if we assume (3) holds. Consider an extreme example of a bimodal density—a density

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[23] We assume, of course, that given the functional form of U, a is constrained so that $0 \leqslant p_i \leqslant \frac{1}{2}$ for all θ and ψ (if $p_i > \frac{1}{2}$, the citizen's over-all probability voting exceeds 1 whenever $\mathbf{x} = \theta = \psi$). For example, suppose the θ varies between 0 and 1, that a is a positive constant, and that $U(\theta/\mathbf{x}) = 1 - (x - \theta)^2$. Then, we must have $0 \leqslant a - a(x - \theta)^2 \leqslant \frac{1}{2}$. If $x = 0$, this condition requires that $0 \leqslant a \leqslant \frac{1}{2}$. In all subsequent examples we assume our constants or functions do not permit probabilities to be less than 0 or greater than 1.

FIGURE 3
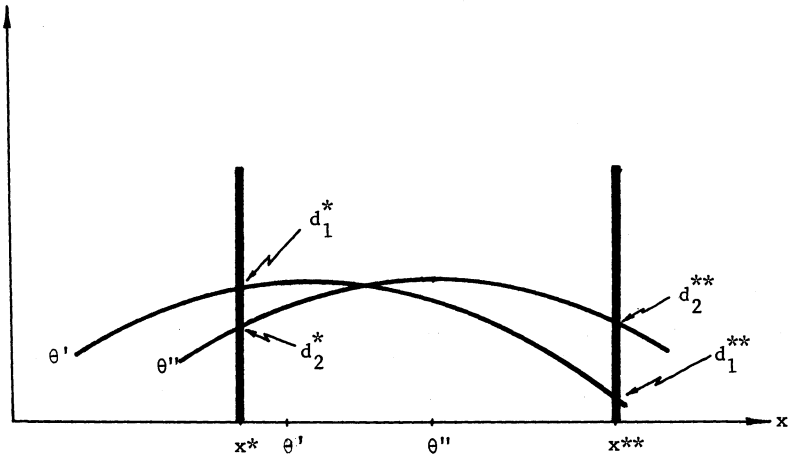
in which citizens prefer either of two distinct policies. Representing this density in Figure 3 by two vertical lines, we graph also two curves for p_1—one for $\theta = \theta'$ and the other for $\theta = \theta''$ such that $\theta' < \theta''$.

Observe now that as candidate one shifts from θ' to θ'' he loses support among citizens preferring x^*, though he gains support among citizens preferring x^{**}. A few pencil exercises reveal, moreover, that the concavity of p_1 guarantees that the rate at which the candidate gains the support of citizens at x^{**} exceeds the rate at which he loses the support of citizens at x^*, that is, $d_2^{**} - d_1^{**} > d_1^* - d_2^*$ (in Figure 3). Hence, both vote-maximizing and plurality-maximizing candidates prefer a strategy closer to the mean than to a mode—they prefer θ'' to θ'.

Contrasting this result with that obtained from formulating alienation in terms of the function g, notice that with g a shift from θ' to θ'' succeeds only in losing the support of citizens at x^*: since $g = 0$ for all citizens whose preferences lie to the right of the midpoint between the candidates' strategies—$(\theta + \psi)/2$—the candidate secures no additional support from citizens at x^{**}. Thus, the candidate prefers the mode of $f(x)$ rather than its mean.

Our assumptions about p_i do not constrain us, though, to formulate a citizen's calculus as (3). First, observe that (3) limits the maximum value of p_i to one-half; otherwise, the citizen's over-all probability of voting, $p_1 + p_2$, exceeds one if $x = \theta = \psi$. To permit larger limits

on p_i, assume that the probability that the citizen votes for candidate one, p_1, is a function of the utility he associates with candidate two's strategy, $U(\psi/x)$, as well as a function of the utility he associates with candidate one's strategy, $U(\theta/x)$. That is,

(4)
$$p_1 = a[U(\theta/x)] - b[U(\psi/x)] + K;$$
$$p_2 = a[U(\psi/x)] - b[U(\theta/x)] + K;$$

(in which $da/dU \geqslant 0$, $db/dU \geqslant 0$, $d^2a/dU^2 \leqslant 0$, $d^2b/dU^2 \leqslant 0$, and $K \geqslant 0$; and again, a and b can be either functions or positive constants). We illustrate p_1 for several values of ψ in Figure 4.

The difference between examples (3) and (4) is that if we assume (3), a citizen's probability of voting for a candidate is a function only of that candidate's strategy. If we assume (4), p_1 again increases as θ approaches the citizen's ideal preference, x; but p_1 is a function of ψ also, since it decreases as ψ approaches x.

Observe now that examples (3) and (4) are not consistent with the hypothesis of abstention because of *indifference* (or cross-pressures) —the hypothesis that a citizen's probability of voting decreases as the utility difference between the candidates' positions decreases. With example (3), the citizen's over-all probability of voting equals

FIGURE 4
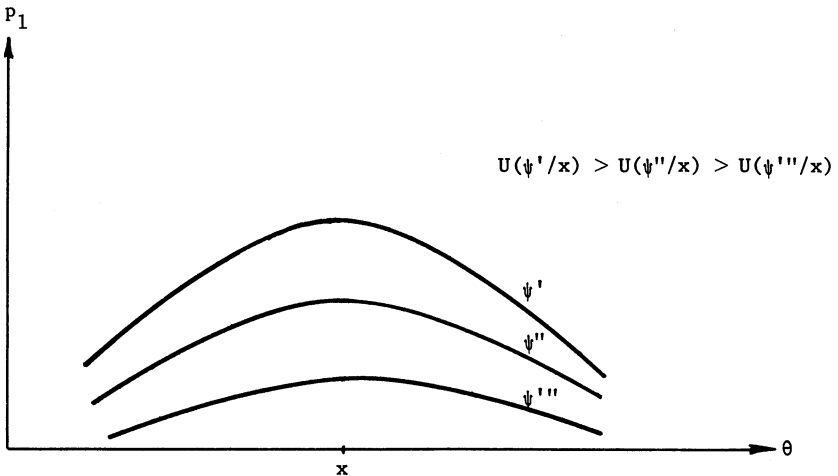
$$U(\psi'/x) > U(\psi''/x) > U(\psi'''/x)$$

$$p_1 + p_2 = a[U(\theta/\mathbf{x}) + U(\psi/\mathbf{x})],$$

whereas with example (4), it equals

$$p_1 + p_2 = (a - b)[U(\theta/\mathbf{x}) + U(\psi/\mathbf{x})]$$

(assuming that a and b are constants). For both examples, $p_1 + p_2$ increases as either $U(\theta/\mathbf{x})$ or $U(\psi/\mathbf{x})$ increases, which is to say that the citizen's probability of voting does not decrease necessarily as the difference between these utilities decreases.

To formulate indifference in our model, observe that examples (3) and (4) are both derived from assumptions A2 and A3. We obtain (3) by deleting the words "decreases or" from A3, in which case our assumptions about p_1 read: "p_1 increases as $U(\theta/\mathbf{x})$ increases and remains unchanged as $U(\psi/\mathbf{x})$ increases." We obtain (4) by deleting the words "or remains unchanged" from A3, in which case our assumptions about p_1 read: "p_1 increases as $U(\theta/\mathbf{x})$ increases and decreases as $U(\psi/\mathbf{x})$ increases." These examples are not consistent with assumption A2', however, because they do not force citizens to have a zero probability of voting for candidates whose positions they do not prefer. Assumption A2' permits such probabilities, and with it we can model the hypothesis of indifference thus:

(5a) $\quad p_1 = b[U(\theta/\mathbf{x}) - U(\psi/\mathbf{x})] \quad$ if $\quad U(\theta/\mathbf{x}) \geqslant U(\psi/\mathbf{x}),$
$\qquad\qquad = 0 \qquad\qquad\qquad\qquad$ otherwise;

(5b) $\quad p_2 = b[U(\psi/\mathbf{x}) - U(\theta/\mathbf{x})] \quad$ if $\quad U(\psi/\mathbf{x}) \geqslant U(\theta/\mathbf{x}),$
$\qquad\qquad = 0 \qquad\qquad\qquad\qquad$ otherwise.

(b is a positive constant). We illustrate this formulation in Figures 5a and 5b. Observe that with example (5), a citizen's probability of voting is either p_1 or p_2 because the citizen votes only for the candidate whose position he prefers. And, since either p_1 or p_2 decreases as the utility difference between the candidates' positions decreases, (5) conforms to indifference.[24]

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[24] We can show that with the appropriate substitutions for U, (5) closely parallels a formulation of indifference considered elsewhere. Specifically, assume that U is the inverse of the quadratic metric and to simplify exposition, that the election is unidimensional. Let

$$U(\theta/\mathbf{x}) = \lambda - \beta(x_1 - \theta_1)^2;$$

FIGURE 5a
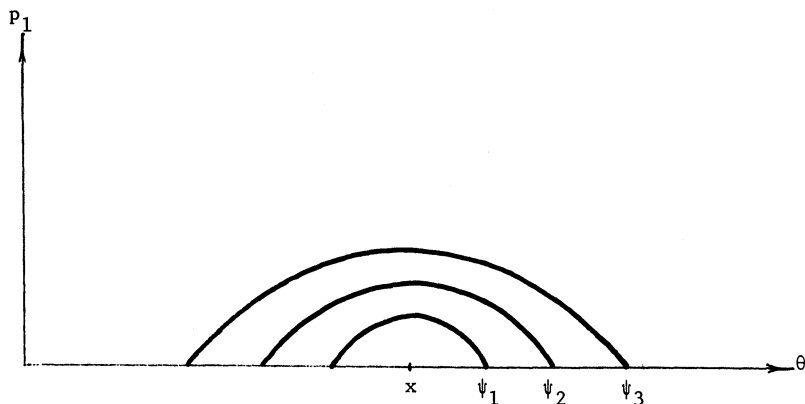
Suppose now that citizens abstain because of both alienation and indifference. We can combine these two hypotheses into a single assumption thus:

$$p_1 = \xi a[U(\theta/x)] + (1 - \xi) b[U(\theta/x) - U(\psi/x)]$$
$$\text{if} \quad U(\theta/x) \geqslant U(\psi/x),$$
$$= \xi a[U(\theta/x)] \qquad \text{otherwise;}$$

(6)

$$p_2 = \xi a[U(\psi/x)] + (1 - \xi) b[U(\psi/x) - U(\theta/x)]$$
$$\text{if} \quad U(\psi/x) \geqslant U(\theta/x),$$
$$= \xi a[U(\psi/x)] \qquad \text{otherwise;}$$

$$U(\psi/x) = \lambda - \beta(x_1 - \psi_1)^2.$$

Assuming that the citizen prefers θ_1 to ψ_1, we substitute these identities into (5a) and obtain,

$$p_1 = -b\beta[\theta_1{}^2 - \psi_1{}^2 + 2x_1\psi_1 - 2x_1\theta_1]$$
$$= -2b\beta[(\theta_1{}^2 - \psi_1{}^2)/2 - x_1(\theta_1 - \psi_1)]$$
$$= 2b\beta[(\theta_1 + \psi_1)/2 - x_1](\psi_1 - \theta_1).$$

The probability p_1 now exhibits two properties. First, since $2b\beta > 0$ by assumption, the citizen's probability of voting for the candidate whose position is θ_1 decreases as $(\psi_1 - \theta_1)$ decreases, *ceteris paribus*. Second, p_1 decreases as the midpoint between the candidates' strategies, $(\theta_1 + \psi_1)/2$, approaches x_1. Except that p_1 is concave in θ_1 and convex in ψ_1, this is identical to the formulation of indifference illustrated by Figure 7 for concave (convex) utility (loss) functions in Ordeshook, "Some Extensions," 55–56.

FIGURE 5b
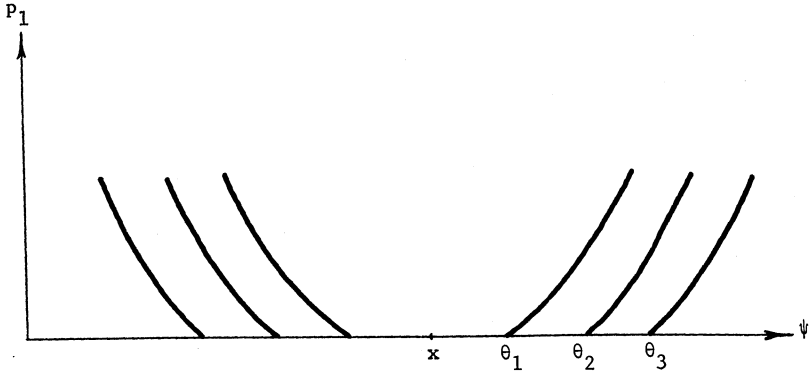
(where $0 \leqslant \xi \leqslant 1$). Hence, if the citizen prefers θ to ψ, his over-all probability of voting equals,

$$p_1 + p_2 = \xi a[U(\theta/x) + U(\psi/x)] + (1 - \xi) b[U(\theta/x) - U(\psi/x)]$$

(letting a be a constant). As the alienation hypothesis suggests, this probability tends to increase as either $U(\theta/x)$ or $U(\psi/x)$ increases (because of the influence of the first term to the right of the equality sign). But, as the indifference hypothesis suggests, this probability decreases as the utility difference between $U(\theta/x)$ and $U(\psi/x)$ decreases (because of the influence of the second term). Furthermore, we can adjust the relative importance of alienation and indifference by varying the magnitude of ξ.

We could continue constructing examples of p_i—perhaps by imagining some relationship between θ and ψ and a citizen's calculus of voting unlike alienation or indifference or by devising some alternative formulations for these two hypotheses. Let us turn, however, to the problem of identifying the location of an electoral equilibrium in our model.

VI

The Location of θ^* *and* ψ^*

To this point our analysis focuses on the problem of ascertaining whether or not an electoral equilibrium exists. We note in our first

section, though, that analyses of the voting paradox and spatial theories confront two problems: (1) ascertaining sufficient conditions for the existence of an equilibrium, and (2) ascertaining the location of the equilibrium. Having concluded our discussion of the first problem, we turn now to the second.

We begin by observing that three distinct approaches to this problem are available. First, we can seek a general method—a numerical procedure—to locate the equilibrium. Second, we can reintroduce the assumptions of previous analyses—assumptions such as symmetric preference distributions and common patterns of saliency—and see if the equilibrium is the mean in our model as it is in other models. Third, we can attempt to generate a new condition that establishes the equilibrium at some particular point such as the mean preference. We consider in this section all three approaches.

Turning first to the construction of a numerical procedure for ascertaining the location of an equilibrium, we observe that unless we are willing to admit additional assumptions about the density of preferences, $U(\theta/x)$, or p_i, general sentences as to the location of equilibrium strategies are unlikely: far too many variables like the nature of citizens' biases are unspecified. Instead, we must approach our task in any election from a perspective closely akin to engineering. That is, we must ascertain the distribution of preferences, the form of citizens' utility functions, the biases of citizens that p_i reflects, and the constraints on the candidates' strategies. Then, by applying suitable numerical procedures we can attempt to calculate the equilibrium solution. (Parenthetically, we note that this approach parallels the analysis of two-person, zero-sum, finite games. In that analysis, the minimax theorem establishes the existence of an equilibrium-strategy pair, but, given the admissible variety of forms of such games, it does not identify the equilibrium. Instead, from the correspondence between these games and linear programming, we know that the Simplex method for solving linear programming problems can be interpreted as a numerical procedure for calculating minimax strategies.)[25] The particular relevance of our theory here is that: first, it identifies the theoretical variables that must be measured; second, it tells us that what we are attempting to calculate exists. Hence, we know that we are not

[25] For a discussion of this correspondence see, for example, Luce and Raiffa, *Games and Decisions*, 408–446; Owen, *Game Theory*, 38–70.

applying our numerical procedures to a fruitless search. Finally, the method we use to prove our theorems in the Appendix to this essay permits us to employ a general numerical procedure for calculating equilibria. Specifically, we know that if the candidates initially are at *any* pair of positions, say θ_0 and ψ_0, and if they alter their strategies in accordance with the rule,

$$
\begin{aligned}
d\theta_i/dt &= a\partial\varphi_1/\partial\theta_i & i &= 1, \ldots, n, a > 0 \\
d\psi_j/dt &= a\partial\varphi_2/\partial\psi_j & j &= 1, \ldots, n, a > 0,
\end{aligned}
$$

(7)

they converge to (θ^*, ψ^*) in the limit.[26] That is, if we permit the candidates to alter their strategies continuously *on each issue* and if the candidates alter their strategies in a way that increases their pluralities at the fastest rate, they converge to the electoral equilibrium.[27] (We refrain at this point from attempting to interpret this result substantively —that is, asserting that candidates might alter their strategies in accordance with (7)—but such an interpretation—speculation—is attractive.) Assuming, then, that our calculations make use of some sort of computer simulation, we can locate our candidates at any initial

[26] A similar expression can be formulated if the candidates maximize their expected vote. For such candidates, however, we must assume that the condition identified in fn. 16 is satisfied. Observe also that we must modify (7) so as to specify an optimal path if one or more of the candidates bumps into the constraints we use in Theorems 3 and 4, that is, if the candidate prefers to violate (2) but is restricted from doing so. Briefly, we assume that the first candidate adjusts his positions as follows (a similar function can be formulated for candidate two):

$$
\begin{aligned}
\frac{d\theta_i}{dt} &= \frac{\partial\varphi_1}{\partial\theta_i} + \sum_{r=1}^{R} \delta_j \frac{\partial G_r}{\partial\theta_i} \\
&= a_i(\theta, \psi, \delta)
\end{aligned}
$$

where δ is chosen to minimize $[\sum_{i=1}^{n} a_i^2]^{1/2}$ subject to

$$
\delta_r \geqslant 0 \qquad \text{if} \quad G_r = 0
$$

Formidable as this condition might appear, it has a simple interpretation. Specifically, if a candidate runs into a constraint, he proceeds along the constraint boundary in the optimal direction until some admissible move of the constraint increases his plurality at a faster rate. If this latter possibility occurs, the condition reduces to (7).

[27] Rosen, "Existence and Uniqueness," 532–533, also offers an approximation to this process which permits the candidates to make discrete rather than continuous moves.

strategy pair (θ_0, ψ_0). We can then let the simulated candidates alter their strategies in accordance with (7) with the guarantee that the candidates converge to the pure strategy equilibrium (θ^*, ψ^*).

It is satisfying to know that a general numerical procedure exists for calculating equilibria, but it is also desirable to search for conditions under which the location of the equilibrium is at some readily identifiable point, such as the electorate's mean preference. We approach this search, first, by considering the multidimensional condition for equilibrium that we reviewed earlier. Specifically, assume that the electorate's distribution of preferences on the issues, $f(\mathbf{x})$, is symmetric on each issue and that citizens who prefer diametrically opposed positions on every issue share a common pattern of saliency for all issues. Our theorem is:

THEOREM 5: *If the conditions of Theorem 1 and Corollary 1.1 or the conditions of Theorem 2 and Corollary 2.1 hold, then, if $f(\mathbf{x})$ is symmetric and if $U(\mathbf{X}/-\mathbf{x}) = U(-\mathbf{X}/\mathbf{x})$ for all $\mathbf{X}$ and $\mathbf{x}$, the unique pure-strategy equilibrium for both plurality- and vote-maximizing candidates is the mean preference of the electorate.*

We cannot regard Theorem 5 as a surprising result. It reaffirms the conclusions of the spatial-analysis theorems that impose symmetry also. It is interesting, then, to see if we can formulate a condition that does not require symmetric densities of preference, thereby permitting us to offer a new condition for the convergence of candidates to the mean. We offer the following theorem:

THEOREM 6: *If the conditions of Theorem 1 and Corollary 1.1 or the conditions of Theorem 2 and Corollary 2.1 hold, then, if $\partial p_1/\partial \theta_j$ and $\partial p_2/\partial \theta_j$ are linear functions of $\mathbf{x}$ for all $\theta = \psi$, the unique pure strategy equilibrium for both plurality- and vote-maximizing candidates is the mean preference of the electorate.*

Unfortunately, unlike a condition such as symmetry, the central condition of Theorem 6 conveys little, if any, substantive meaning. Clearly, it is difficult to see what constraints on a citizen's utility function and on p_i we require before $\partial p_i/\partial \theta_j$ is assured of being a linear function of $\mathbf{x}$. Instead of providing an interpretation in the body of this

article, we have judged it best to relegate the explanation to a nota.[28]

<div align="center">CONCLUSIONS</div>

Our primary objective in this essay is to formulate a new class of sufficient conditions for the existence of a majority preference and then to apply these conditions to a spatial analysis of electoral competition. By focusing on a citizen's calculus of voting rather than on the distribution of preferences or patterns of issue saliency, and by assuming that the postulate of rational choice leads to probabilistic rather than deterministic decisions, we construct a theory that either eliminates or weakens several assumptions other spatial analyses of electoral equilibrium employ. Our theorems and corollaries do not, of course, render these prior analyses obsolete. Our theory, for example, does not permit everyone to vote if citizens choose deterministically. We must continue, then, to regard the analysis of Black et al. as an admissible base upon which to construct a spatial model. Nor does every plausible formulation of a citizen's calculus satisfy our assumptions: the function g, which we illustrate in Figure 1, for example, fails to satisfy A4.

---

[28] The function $w_1(x) = a_0 + a_1x$ illustrates a linear function of x. That is, $w_1(x)$ does not contain any term that involves powers of x greater than 1. Alternatively, $w_2(x) = a_0 + a_1x + a_2x^2$ is not a linear function because it contains the term a_2x^2. To ascertain whether or not the conditions of Theorem 6 are satisfied, then, we must know something about the functional forms of p_i and U. Suppose p_i satisfied (3), where a is a positive constant. Additionally, assume that each citizen's utility function is the inverse of the quadratic metric. Thus,

$$p_1 = a\lambda - a\beta[(x_1 - \theta_1)^2 + \cdots + (x_j - \theta_j)^2 + \cdots].$$

Differentiating p_1 with respect to θ_j yields

$$\frac{\partial p_1}{\partial \theta_j} = 2a\beta(x_j - \theta_j) = -2a\beta\theta_j + 2a\beta x_j$$

By letting $w_1(x) = \partial p_1/\partial \theta_j$, $a_0 = -2a\beta$, we can readily see that $\partial p_1/\partial \theta_j$ is a linear function of x_j. Hence, if all citizens vote according to (3) and if the inverse quadratic metric characterizes their utility functions (note however that a and β cannot be functions of x which is to say that all citizens must weight the relative saliencies of the issues in an identical fashion), the conditions of Theorem 6 are satisfied. Thus, we have one intuitively appealing condition under which the candidates converge to the electorate's mean preference even though $f(x)$ is not symmetric. We must note, however, that examples (4) and (5) do not satisfy the conditions of the theorem.

Hence, we must regard an analysis that utilizes such a function as an alternative to our theory. This point deserves special emphasis since, with g, vote-maximizing candidates may adopt distinct strategies while, with our assumptions, such candidates converge.

Clearly, then, we have several assumptions to choose from in spatial theory. Frequently, alternative combinations of assumptions yield equivalent theorems, such as the proof that the mean is the social choice if $f(x)$ is a symmetric, unimodal density and if the candidates maximize plurality. But for different situations, as when the candidates maximize votes, the choice of an assumption can be critical. Some empirical evidence exists already that suggests that our model of citizens—specifically, examples (3) and (4) and the assumption that citizens retain a positive probability of voting for candidates that they do not prefer spatially—is a legitimate base upon which to proceed with the task of constructing spatial theory.[29] It is our hope, of course, that empirical research will be directed further to the task of ascertaining the appropriateness of one assumption or another.

Perhaps the most important conclusion of this essay, however, concerns the existence of equilibrium in political processes. Because it is generally conceded that the unidimensional and multidimensional conditions for equilibrium of Black et al. are highly restrictive, the question arises as to whether political processes—especially those that derive from a voting procedure such as majority rule—ever attain equilibrium. If they do not, or if our abstract descriptions of them do not, serious doubts exist as to our ability to offer parsimonious theories that explain and predict events. Our analysis with a probabilistic calculus of choice suggests, however, that disequilibrium may be a less pervasive phenomenon than is otherwise believed.

[29] Rosenthal, "Electoral Participation," in press.

APPENDIX

The proofs of Theorems 1 and 2 consist essentially of showing that if our assumptions about p_i are satisfied for all citizens, then φ_i satisfies (i) and (ii). These theorems, the numerical procedure for calculating equilibrium strategies, and Theorems 3 and 4 also follow directly from J. B. Rosen's results. The corollaries to Theorems 1 through 4 as well as Theorems 5 and 6 are proved directly. To simplify exposition, however, we delete from this appendix those exact mathematical details which, while important in any completely rigorous proof, necessitate a rather lengthy discussion of some definitions of real variable analysis.[30]

We begin by altering our notation slightly; we add the superscript c to p_i and $\mathbf{x}$ so as to denote a particular citizen in the set C of all citizens. Observe now that the expected number of votes for candidates 1 and 2 are expressed respectively as,

$$(1A) \qquad V_1(\theta, \psi) = \sum_{c \in C} p_1{}^c(U(\theta/\mathbf{x}^c), U(\psi/\mathbf{x}^c))$$

$$(2A) \qquad V_2(\theta, \psi) = \sum_{c \in C} p_2{}^c(U(\theta/\mathbf{x}^c), U(\psi/\mathbf{x}^c))$$

Hence, candidate one's expected vote is obtained by ascertaining each citizen's probability of voting for θ and then by summing these probabilities over the entire electorate.

Turning now to Theorem 1, we assume that the candidates maximize expected plurality. Thus,

$$(3A) \qquad \varphi_1(\theta, \psi) = V_1(\theta, \psi) - V_2(\theta, \psi)$$

$$(4A) \qquad \varphi_2(\theta, \psi) = V_2(\theta, \psi) - V_1(\theta, \psi)$$

To prove Theorem 1 using the general game theoretic result about two-person zero-sum, infinite games, we must show that $\varphi_1(\theta, \psi)$ is concave in θ and convex in ψ, and that $\varphi_2(\theta, \psi)$ is concave in ψ and convex in θ. (It is readily seen that φ_1 and φ_2 are continuous in θ and ψ). We use the following general results about convex (concave) functions:

---

[30] Briefly, these conditions are: (1) X, the space of all admissible strategies and preferences, is a convex compact subset of R^n; (2) U is a continuously differentiable function of θ, and is strictly concave for some subset of C, the set of all citizens; (3) $p_1{}^c - p_2{}^c$ is a continuously differentiable function of $U(\theta, \mathbf{x}^c)$ and $U(\psi, \mathbf{x}^c)$, and if A2, $p_i{}^c$ is continuously differentiable and; (4) the convexity and concavity properties of $p_1{}^c$ and $p_2{}^c$ are *strict* for some subset of C. To understand why these assumptions are necessary and how they are used in our analysis, see the fully rigorous proof of Theorem 1 and its corollary with assumption A2' in Hinich, Ledyard, and Ordeshook, "Non-Voting and the Existence of Equilibrium Under Majority Rule," *Journal of Economic Theory* 4 (March 1972), 144–153.

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(I): If $r(x)$ is convex (concave) in x, $-r(x)$ is concave (convex) in x.

(II): If $r_i(x)$ is convex (concave) in x for all i, then $R(x) = \Sigma_i\, r_i(x)$ is convex (concave) in x.

(III): If $t(x)$ is convex (concave) in x, and if $r(t(x))$ is convex (concave) in t, then r is convex (concave) in x where $r' \geq 0$.

From A4, $p_2{}^c$ is concave in $U(\psi/x^c)$ and convex in $U(\theta/x^c)$ so that from (I), $-p_2{}^c$ is concave in $U(\theta/x^c)$ and convex in $U(\psi/x^c)$. From the assumption that $p_1{}^c$ is concave in $U(\theta/x^c)$ and convex in $U(\psi/x^c)$ and from (II), then $p_1{}^c - p_2{}^c$ is concave in $U(\theta/x^c)$ and convex in $U(\psi/x^c)$. Candidate one's plurality, however, is simply a sum of these functions so that from (II), φ_1 is concave in $U(\theta/x^c)$ and convex in $U(\psi/x^c)$. Finally, from the assumption that U is a concave function of its argument—assumption A1— and from (III), it follows that φ_1 is concave in θ and convex in ψ. Also, since $\varphi_2 = -\varphi_1$, it follows from (I) that φ_2 is concave in ψ and convex in θ.

We can also show that Theorem 1 as well as the numerical procedure for calculating equilibrium strategies, follow from Rosen's results. First, we must show that the matrix

$$(5A) \qquad \mathbf{Q}(\theta,\, \psi) = \begin{pmatrix} 2\nabla_{\theta\theta}\varphi_1 & \nabla_{\theta\psi}\varphi_1 + \nabla_{\theta\psi}\varphi_2 \\ \nabla_{\psi\theta}\varphi_1 + \nabla_{\psi\theta}\varphi_2 & 2\nabla_{\psi\psi}\varphi_2 \end{pmatrix}$$

is negative definite where $\nabla_{\theta\theta}\varphi_k$, $\nabla_{\theta\psi}\varphi_k$, $\nabla_{\psi\theta}\varphi_k$, and $\nabla_{\psi\psi}\varphi_k$ are the matrices $(\partial^2\varphi_k/\partial\theta_i\,\partial\theta_j)$, $(\partial^2\varphi_k/\partial\theta_i\,\partial\psi_j)$, $(\partial^2\varphi_k/\partial\psi_i\,\partial\theta_j)$, and $(\partial^2\varphi_k/\partial\psi_i\,\partial\psi_j)$ respectively for $k = 1, 2$. Note that since $\varphi_1 = -\varphi_2$, (5A) can be written,

$$(6A) \qquad \mathbf{Q}(\theta,\, \psi) = 2 \begin{pmatrix} \nabla_{\theta\theta}\varphi_1 & 0 \\ 0 & -\nabla_{\psi\psi}\varphi_1 \end{pmatrix}$$

If, for example, the election involves a single issue ($n = 1$), (6A) becomes,

$$\begin{pmatrix} \dfrac{\partial^2\varphi_1}{\partial\theta_1{}^2} & 0 \\ 0 & -\dfrac{\partial^2\varphi_1}{\partial\psi_1{}^2} \end{pmatrix}$$

Hence, if $\mathbf{Q}$ is negative definite, the second derivative of φ_1 with respect to ψ_1 is positive. That is, if $\varphi_1[\varphi_2]$ is a concave function of $\theta_1[\psi_1]$ and a convex function of $\psi_1[\theta_1]$, $\mathbf{Q}(\theta_1,\, \psi_1)$ is negative definite in our one dimensional illustration.

To prove that $\mathbf{Q}$ is negative definite for more than one dimension we define the function,

$$(7A) \qquad s^c(U(\theta/x^c),\, U(\psi/x^c)) = p_1{}^c(U(\theta/x^c),\, U(\psi/x^c)) - p_2{}^c(U(\theta/x^c),\, U(\psi/x^c))$$

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From (1A) and (3A), then,

(8A) $$\varphi_1(\theta, \psi) = \sum_{c \in C} s^c$$

Clearly, now, if $p_i{}^c$ satisfies A2 through A4 or A2', A3 and 4,

(9A) $$\frac{\partial s^c}{\partial U(\theta/x^c)} \geqslant 0$$

(10A) $$\frac{\partial s^c}{\partial U(\psi/x^c)} \leqslant 0$$

(11A) $$\frac{\partial^2 s^c}{\partial U(\theta/x^c)^2} \leqslant 0$$

(12A) $$\frac{\partial^2 s^c}{\partial U(\psi/x^c)^2} \geqslant 0$$

with at least the first inequality strict for some c.

Let $\nabla_\theta U(\theta/x^c) = (\partial U/\partial \theta_1, \ldots, \partial U/\partial \theta_n)$, and let $\nabla_{\theta\theta} U(\theta/x^c)$ be the matrix $(\partial^2 U/\partial \theta_i \partial \theta_j)$ which is negative definite by the concavity of U in θ. Finally, let $\nabla_{\theta\theta} s^c$ be the matrix $(\partial^2 s^c/\partial \theta_i \partial \theta_j)$. Observe that,

(13A) $$\nabla_{\theta\theta} s^c = \left(\frac{\partial^2 s^c}{[\partial U(\theta/x^c)]^2} \right) (\nabla_\theta U)(\nabla_\theta U)' + \left(\frac{\partial s^c}{\partial U(\theta/x^c)} \right) \nabla_{\theta\theta} U$$

For example, if $n = 1$, (13) reduces to,

$$\frac{\partial^2 s^c}{\partial \theta_1{}^2} = \frac{\partial^2 s^c}{[\partial U(\theta_1/x_1{}^c)]^2} \cdot \frac{\partial U(\theta_1/x_1{}^c)}{\partial \theta_1} + \frac{\partial s^c}{\partial U(\theta_1/x_1{}^c)} \cdot \frac{\partial^2 U(\theta_1/x^c{}_1)}{\partial \theta_1{}^2}$$

which is simply what we obtain if we calculate the second derivative of s^c directly.

From (9A), (11A), and the fact that $\nabla_{\theta\theta} U(\theta/x^c)$ is negative definite, it follows that $\nabla_{\theta\theta} s^c$ is negative definite. Since this is true for all $c \in C$,

$$\nabla_{\theta\theta} \varphi_1 = \sum_{c \in C} \nabla_{\theta\theta} s^c$$

is negative definite. In a similar way, we can show that

$$-\nabla_{\psi\psi} \varphi_1 = \sum_{c \in C} \nabla_{\psi\psi} s^c$$

is negative definite. Thus, from (6A), $Q(\theta, \psi)$ is negative definite. Since φ_1 is concave

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in θ and φ_2 is concave in ψ, Theorem 1 and expression (7) follow directly from Theorems 1, 2, 6, and 9 of Rosen.

To prove Corollary 1.1 (that is, that $\theta* = \psi* = \alpha$), we note that if p_1^c and p_2^c are of identical functional form, $\varphi_1(X, X) = 0$. Suppose now that $\theta* \neq \psi*$ (that is, the candidates do not converge). Since

$$\varphi_1(\theta*, \psi*) \geqslant \varphi_1(\psi*, \psi*) = 0$$

and

$$\varphi_2(\theta*, \psi*) \geqslant \varphi_2(\theta*, \theta*) = 0$$

it follows from $\varphi_1 = -\varphi_2$ that

$$\varphi_1(\theta*, \psi*) = \varphi_2(\theta*, \psi*) = 0$$

Hence, by the (strict) concavity of p_1 in θ.

$$\varphi_1(\xi\theta* + (1 - \xi)\,\psi*, \psi*) > \xi\varphi_1(\theta*, \psi*) + (1 - \xi)\,\varphi_1(\psi*, \psi*) = 0 = \varphi_1(\theta*, \psi*)$$

which is to say that the strategy $\xi\theta* + (1 - \xi)\,\psi*$ yields candidate one a positive plurality if candidate two adopts his "optimal" strategy. Thus, $(\theta*, \psi*)$ is not an equilibrium unless $\theta* = \psi*$. By the concavity of the objective functions, the zero-sum nature of the election, and the nature of the equilibrium, it follows that $\varphi_1(\alpha, \psi) > 0$ if $\psi \neq \alpha$ and $\varphi_2(\theta, \alpha) > 0$ if $\theta \neq \alpha$. The strategy α, then is dominant.

Q.E.D.

Like Theorem 1, Theorem 2 can also be shown to follow directly from those results in game theory pertaining to games with continuous, convex payoffs. As with Theorem 1, however, we again use Rosen's results to establish the validity of the numerical procedure defined by expression (7). First, we must show that the matrix

$$Q_0(\theta, \psi) = \begin{pmatrix} 2\nabla_{\theta\theta}V_1 & \nabla_{\theta\psi}V_1 + \nabla_{\theta\psi}V_2 \\ \nabla_{\psi\theta}V_2 + \nabla_{\psi\theta}V_1 & 2\nabla_{\psi\psi}V_2 \end{pmatrix}$$

is negative definite where $\nabla_{\theta\theta}V_k$, $\nabla_{\theta\psi}V_k$, $\nabla_{\psi\theta}V_k$, and $\nabla_{\psi\psi}V_k$ are the matrices $(\partial^2 V_k/\partial\theta_i\ \partial\theta_j)$, $(\partial^2 V_k/\partial\theta_i\ \partial\psi_j)$, $(\partial^2 V_k/\partial\psi_i\ \partial\theta_j)$, and $(\partial^2 V_k/\partial\psi_i\ \partial\psi_j)$ respectively for $k = 1, 2$.

Suppose now that $\nabla_{\theta\psi}V_k = \nabla_{\psi\theta}V_k = 0$ for $k = 1, 2$. Verbally, suppose that the rate of change in a candidate's total is independent of his opponent's strategy (as in all of our examples except (5)). Then $Q_0(\theta, \psi)$ reduces to,

$$Q_0*(\theta, \psi) = 2 \begin{pmatrix} \nabla_{\theta\theta}V_1 & 0 \\ 0 & \nabla_{\psi\psi}V_2 \end{pmatrix}$$

<center>APPENDIX</center>

<center>(CONTINUED)</center>

By definition,

$$2\nabla_{\theta\theta}V_1 = 2\sum_{c\in C}\left\{\frac{\partial^2 p_1{}^c}{[\partial U(\theta/x^c)]^2}(\nabla_\theta U)(\nabla_\theta U') + \frac{\partial p_1{}^c}{\partial U(\theta/x^c)}(\nabla_{\theta\theta}U)\right\}$$

Since $p_1{}^c$ is concave in $U(\theta/x^c)$, $\partial^2 p_c{}^1/[\partial U(\theta/x^c)]^2 < 0$.

Also, we know that $[\partial U(\theta/x^c)/\partial\theta]^2 \geqslant 0$. It follows then that the first term inside the summation sign is negative definite. Similarly, $\partial p_1{}^c/\partial U(\theta/x^c) > 0$ from A2, but $\nabla_{\theta\theta}U$ is negative definite since U is concave. Hence, the second term is also negative definite. Thus, $2\nabla_{\theta\theta}V_1$ is the sum of negative definite matrices and, therefore, is itself negative definite. In a similar fashion, $2\nabla_{\psi\psi}V_2$ can be shown to be negative definite so that $Q_0{}^*$ is negative definite.

If $\nabla_{\theta\psi}V_k$ and $\nabla_{\psi\theta}V_k$ are not equal to zero, however, then we must consider $Q_0 = A + B$, where

$$A = \begin{pmatrix} \nabla_{\theta\theta}V_1 & 0 \\ 0 & \nabla_{\psi\psi}V_2 \end{pmatrix}$$

$$B = \begin{pmatrix} \nabla_{\theta\theta}V_1 & \nabla_{\theta\psi}V_1 + \nabla_{\theta\psi}V_2 \\ \nabla_{\psi\theta}V_1 + \nabla_{\psi\theta}V_2 & \nabla_{\psi\psi}V_1 \end{pmatrix}$$

Since $A = Q_0{}^*/2$, A is negative definite. Hence, all we must show is that B is negative semidefinite to obtain our result. That is, we must show that

(14A) $yBy' \leqslant 0$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and where y_1 and y_2 are $1 \times n$ nonzero vectors. From the definitions of $\nabla_{\theta\theta}V_1$, and the like, and by straightforward manipulation, it can be shown that (14A) is satisfied whenever

$$\frac{\partial^2 p_1{}^c}{[\partial U(\theta/x^c)]^2} \cdot \frac{\partial^2 p_2{}^c}{[\partial U(\psi/x^c)]^2} \geqslant \left[\frac{\partial^2 p_1{}^c}{\partial U(\theta/x^c)\,\partial U(\psi/x^c)} + \frac{\partial^2 p_2{}^c}{\partial U(\theta/x^c)\,\partial U(\psi/x^c)}\right]^2$$

This is the condition we assume in footnote 16. It is satisfied by all of our examples in Section 5 which pertain to vote-maximizing candidates.

To prove Corollary 2.1, assume that $\theta^* \neq \psi^*$. If $p_1{}^c$ and $p_2{}^c$ are of identical functional form, the candidates' labels are interchangeable, so if (θ^*, ψ^*) is an equilibrium (ψ^*, θ^*) is an equilibrium. It follows then that $V_1(\psi^*, \psi^*) = V_1(\theta^*, \psi^*)$. Now, from the (strict) concavity of V_1 we get,

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(CONTINUED)

$$V_1(\xi\theta^* + (1 - \xi)\,\psi^*, \psi^*) > \xi V_1(\theta^*, \psi^*) + (1 - \xi)\,V_1(\psi^*, \psi^*)$$

or equivalently

$$V_1(\xi\theta^* + (1 - \xi)\,\psi^*, \psi^*) > V_1(\theta^*, \psi^*)$$

which is to say that the strategy $\xi\theta^* + (1 - \xi)\,\psi^*$ is better against ψ^* than is θ^*. Thus, it must be the case that $\theta^* = \psi^*$.

Theorems 3 and 4 are essentially corollaries to Theorems 1 and 2 and they follow in the same way from Rosen's theorems. To prove Corollaries 3.1 and 4.1 we assume again that $\theta^* \neq \psi^*$. Since (ψ^*, ψ^*) is feasible, it can be compared with (θ^*, ψ^*). Applying the appropriate parts of the proofs of Theorems 1 and 2, it follows that (θ^*, ψ^*) is an equilibrium only if $\theta^* = \psi^*$.

To prove Theorems 4 and 5 we assume that if two citizens share the same ideal preference, their utility functions are of the same functional form and their probability of voting functions are of the same functional form. This assumption is not necessary, but our notation is greatly simplified with it. Denoting the distribution of ideal preferences by $f(\mathbf{x})$, now we write

(15A) $$V_1(\theta, \psi) = \sum_{\mathbf{x}\in\mathbf{X}} p_1(U(\theta/\mathbf{x}), U(\psi/\mathbf{x}))\, f(\mathbf{x})$$

Without loss of generality, let $E(\mathbf{x}) = 0$. Observe that from (15A) and from the conditions of Theorem 5,

$$V_1(\theta, \psi) = \sum_{\mathbf{x}\in\mathbf{X}} p_1(U(-\theta/-\mathbf{x}), U(-\psi/-\mathbf{x}))\, f(\mathbf{x})$$

$$= \sum_{\mathbf{x}\in\mathbf{X}} p_1(U(-\theta/-\mathbf{x}), U(-\psi/-\mathbf{x}))\, f(-\mathbf{x})$$

$$= \sum_{\mathbf{x}\in\mathbf{X}} p_1(U(-\theta/\mathbf{x}), U(-\psi/\mathbf{x}))\, f(\mathbf{x})$$

$$= V_1(-\theta, -\psi).$$

Similarly, $V_2(-\theta, -\psi) = V_2(\theta, \psi)$ so that both V_1 and V_2 are radially symmetric about 0. Now assume that $\alpha \neq 0$, where α is the equilibrium. By the (strict) concavity of φ_1 in θ,

$$0 = \varphi_1(0, 0) = \varphi_1(\alpha/2 - \alpha/2, 0) > \varphi_1(\alpha, 0)/2 + \varphi_1(-\alpha, 0)/2$$

But φ_1 is radially symmetric if V_1 and V_2 are radially symmetric so that $\varphi_1(\alpha, 0) = \varphi_1(-\alpha, 0)$. Thus, $\varphi_1(\alpha, 0) < \varphi_1(\alpha, \alpha) = 0$. But (α, α) is the equilibrium and consequently, $\varphi_1(\alpha, 0) > \varphi_1(\alpha, \alpha)$. Thus, $\alpha = 0$.

Appendix

(Continued)

To prove Theorem 6, we observe that at the equilibrium α, $\partial\varphi_1/\partial\theta_i = 0$ and $\partial\varphi_2/\partial\psi_i = 0$ for $i = 1, \ldots, n$ (Rosen, p. 524). Now

$$\frac{\partial\varphi_1}{\partial\theta_i} = \sum_{x \in X} \frac{\partial s}{\partial\theta_i}$$

Let

$$\bar{x} = \sum_x x f(x).$$

By the assumption of linearity, for $\theta = \psi = \alpha$,

$$\frac{\partial\varphi_1}{\partial\theta_i} = \frac{\partial s(U(\theta/\bar{x}), U(\psi/\bar{x}))}{\partial\theta_i}$$

But

$$\frac{\partial s}{\partial\theta_i} = \frac{\partial s}{\partial U(\theta/\bar{x})} \cdot \frac{\partial U(\theta/\bar{x})}{\partial\theta_i} = \frac{\partial\varphi_1}{\partial\theta_i}$$

From (9A), however, we know that $\partial s/\partial U(\theta/x) > 0$. Thus, $\partial\varphi_1/\partial\theta_i = \partial s/\partial\theta_i = 0$ only if $\partial U(\theta/\bar{x})/\partial\theta_i = 0$. Now $U(\theta/\bar{x})$ has a unique maximum at $\theta = \bar{x}$ and thus $\partial U(\theta/\bar{x})/\partial\theta_i = 0$ for $i = 1, \ldots, n$ if and only if $\theta = \bar{x}$. That is, α is the equilibrium if and only if it equals the mean of $f(x)$.